

REFLECTION GROUPS OF THE QUADRATIC FORM

$$-5x_0^2 + x_1^2 + \dots + x_n^2$$

ALICE MARK

Let V be a an $(n+1)$ -dimensional real vector space with basis v_0, \dots, v_n , and let L be the integer lattice generated by the same basis. Given a quadratic form of the form

$$f(x) = -px_0^2 + x_1^2 + \dots + x_n^2$$

with p a positive integer, the group Θ of integral automorphisms is the group of symmetries of L preserving that form and mapping each connected component of the set $\{x : f(x) < 0\}$ to itself. This group splits as a semidirect product

$$\Theta = \Gamma \rtimes H$$

where Γ is generated by reflections and H is a group of symmetries of an associated polytope in hyperbolic n -space [1]. We say L is reflective if H is a finite group. For a fixed p , we wish to know for which values of n the lattice L is reflective. Vinberg answered that question for $p = 1, 2$ in [1] and [3], and McLeod answered it for $p = 3$ in [2]. Here we answer it for $p = 5$.

First we find generators for Γ using Vinberg's algorithm. Then we show that the algorithm terminates after not very many steps when $2 \leq n \leq 8$. Finally we show that when $n \geq 9$, Γ is not finitely generated.

Descriptions of the algorithm abound (see [1] and [2] and others), so I won't give a complete one here. Since the algorithm finds roots, the vectors $e_i = \sum k_j v_j$ it finds are subject to the crystallographic condition. In the case $p = 5$ that means $(e_i, e_i) = 1, 2, 5$, or 10 , and if $(e_i, e_i) = 5$ or 10 then $5 \nmid k_0$ and $5 | k_j$ for $j \neq 0$.

Proposition 1. *The first several vectors that Vinberg's algorithm produces are listed in Table 1.*

$\frac{k_0^2}{(e_i, e_i)}$	e_i	(e_i, e_i)	i	n
$\frac{1}{2}$	$v_0 + 2v_1 + v_2 + v_3 + v_4$	2	$n+3$	≥ 4
	$v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7$	2	$n+4$	≥ 7
$\frac{4}{5}$	$2v_0 + 5v_1$	5	$n+1$	≥ 2
$\frac{1}{1}$	$v_0 + 2v_1 + v_2 + v_3$	1	$n+3$	3
	$v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6$	1	$n+4$	6
$\frac{9}{5}$	$3v_0 + 5v_1 + 5v_2$	5	$n+2$	≥ 2

TABLE 1. Vectors found with Vinberg's Algorithm. The labels i are chosen for convenience to later arguments rather than the order in which the algorithm finds them.

Proof. The batches labeled $\frac{1}{10}, \frac{1}{5}$, and $\frac{4}{10}$ are empty because $(e, e) = 5$ or 10 , and there is no way to write $(e, e) + 5k_0^2 = 15, 20$ or 30 as a sum of squares of integers all divisible by 5 .

The batch labeled $\frac{1}{2}$ consists of vectors $e = \sum_{i=0}^n k_n v_n$ where

$$\sum_{i=1}^n k_i^2 = (e, e) + 5k_0^2 = 7$$

7 may be written as a sum of squares in two ways. This batch contains one vector if $n \geq 4$, and two if $n \geq 7$. These two vectors have inner product 0 , so we keep both of them.

The batch labeled $\frac{4}{5}$ consists of vectors $e = \sum_{i=0}^n k_i v_i$ where

$$\sum_{i=1}^n k_i^2 = (e, e) + 5k_0^2 = 25$$

and $5|k_i$ for all $i > 0$. The vector $2v_0 + 5v_1$ has inner product 0 and -5 with the vectors in the previous nonempty batch so we keep it for all $n \geq 2$.

The batch labeled $\frac{9}{10}$ is empty since $(e, e) = 10$ and $(e, e) + 5k_0^2 = 55$ cannot be written as a sum of squares of integers all divisible by 5 .

The batch labeled $\frac{1}{1}$ consists of vectors $e = \sum_{i=0}^n k_i v_i$ where

$$\sum_{i=1}^n k_i^2 = (e, e) + 5k_0^2 = 6$$

There are two ways to write 6 as a sum of squares. One of these produces the vector $v_0 + 2v_1 + v_2 + v_3$. This has inner product 0 with the vector in batch $\frac{4}{5}$, so we keep it when $n = 3$, but it has positive inner product with the vector $v_0 + 2v_1 + v_2 + v_3 + v_4$ in batch $\frac{1}{2}$, so we throw it out when $n \geq 4$.

The other way to write 6 as a sum of squares produces the vector $v_0 + v_2 + v_3 + v_4 + v_5 + v_6$. This has inner product 0 with the vector $v_0 + 2v_1 + v_2 + v_3 + v_4$ in batch $\frac{1}{2}$ and inner product -5 with the vector in batch $\frac{4}{5}$, so we keep it when $n = 6$. It has positive inner product with the vector $v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7$ in batch $\frac{1}{2}$, so we throw it out when $n \geq 7$.

The batch labeled $\frac{16}{10}$ is empty since $(e, e) = 10$ and $(e, e) + 5k_0^2 = 90$ cannot be written as a sum of squares of integers all divisible by 5 .

The batch labeled $\frac{9}{5}$ consists of vectors $e = \sum_{i=0}^n k_i v_i$ where

$$\sum_{i=1}^n k_i^2 = (e, e) + 5k_0^2 = 50$$

and $5|k_i$ for all $i > 0$. There is exactly one vector satisfying this. $3v_0 + 5v_1 + 5v_2$ has non-positive inner product with all the vectors in previous batches, so we keep it for all $n \geq 2$.

□

Proposition 2. *The diagrams in figure 1 all describe acute angled polytopes of finite volume.*

We will prove this using Vinberg's criterion for finite volume. Our notation follows [1], and the criterion we use is Proposition 2 in that paper. A list of the affine Coxeter diagrams can also be found there in Table 2.

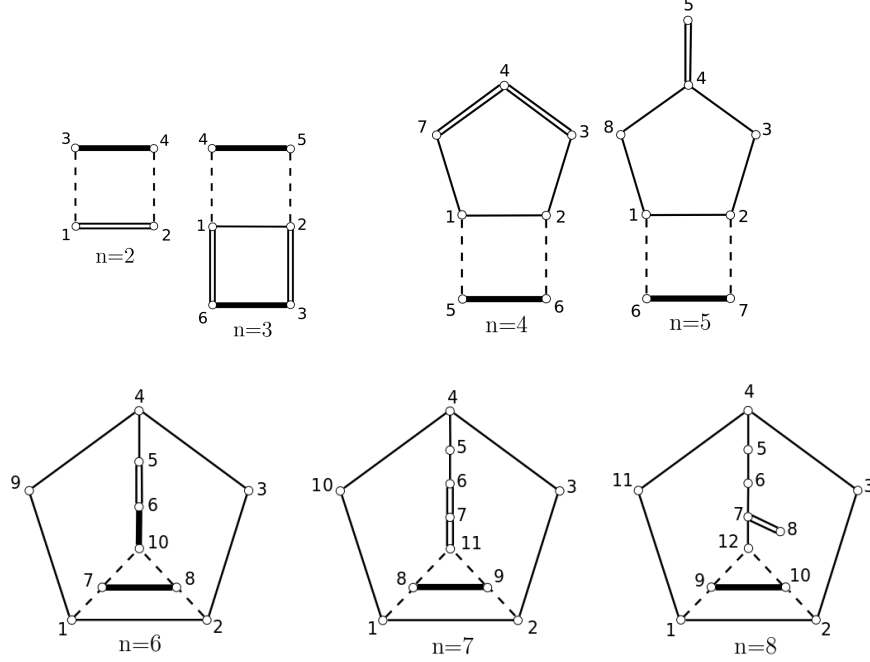


FIGURE 1. Coxeter diagrams for the hyperbolic reflection groups associated to the form $-5x_0^2 + x_1^2 + \dots + x_n^2$

If conditions 1 and 2 of the finite volume criterion are met, then any facet of the polyhedral angle K_S that passes out of the cone \mathfrak{C} intersects the boundary of the cone in a line in V corresponding to a cusp of the polytope.

The only cocompact hyperbolic diagrams occurring as subdiagrams of the graphs in figure 1 are dotted line edges, and the only maximal affine subdiagrams have rank $n - 1$.

Listed in table 2 are the index sets for the affine subdiagrams and the types of their maximal extensions, which shows that condition 1 of the finite volume criterion is satisfied for all of the diagrams in figure 1.

It remains to show that condition 2 of the finite volume criterion is met. There are 4 different dotted line components which appear. We will show that each satisfies condition 2.

A sufficient condition for a dotted line subgraph to satisfy condition 2 is that there exist a set of vertices $T \subset I$ such that the diagram with vertex set T is a spherical subdiagram of rank at least $n - 1$, and there are no edges between the vertices of S and the vertices of T (this is a corollary of Proposition 2 in [1]). To see why it is true, compute the dimension of $(S \cup T)^\perp$.

The two dotted line subgraphs that appear only when $n = 6, 7, 8$ can be shown to satisfy condition 2 using this fact. The first of these has vertex set $S = \{n+2, n+3\}$. Let $T = \{2, 3, \dots, n-1, n+1\}$. When $n = 6$, T has type D_5 . When $n = 7$, T has type E_6 . When $n = 8$, T has type E_7 . In each case this is a spherical diagram of rank $n - 1$ with no edges joining any vertex in T to any vertex in S .

n	index set	maximal diagram type
2	$\{3, 4\}$	\tilde{A}_1
3	$\left. \begin{array}{l} \{3, 6\} \\ \{4, 5\} \end{array} \right\}$	\tilde{A}_1^2
4	$\left. \begin{array}{l} \{3, 4, 7\} \\ \{5, 6\} \end{array} \right\}$	$\tilde{A}_1 \tilde{C}_2$
5	$\left. \begin{array}{l} \{1, 2, 3, 4, 8\} \\ \{3, 4, 5, 8\} \\ \{6, 7\} \end{array} \right\}$	\tilde{A}_4 $\tilde{A}_1 \tilde{B}_3$
6	$\left. \begin{array}{l} \{1, 2, 3, 4, 9\} \\ \{6, 10\} \\ \{3, 4, 5, 6, 9\} \\ \{7, 8\} \end{array} \right\}$	$\tilde{A}_1 \tilde{A}_4$ $\tilde{A}_1 \tilde{B}_4$
7	$\left. \begin{array}{l} \{1, 2, 3, 4, 10\} \\ \{6, 7, 11\} \\ \{3, 4, 5, 6, 7, 10\} \\ \{8, 9\} \end{array} \right\}$	$\tilde{A}_4 \tilde{C}_2$ $\tilde{A}_1 \tilde{B}_5$
8	$\left. \begin{array}{l} \{1, 2, 3, 4, 11\} \\ \{6, 7, 8, 12\} \\ \{3, 4, 5, 6, 7, 8, 11\} \\ \{9, 10\} \end{array} \right\}$	$\tilde{A}_4 \tilde{B}_3$ $\tilde{A}_1 \tilde{B}_6$

TABLE 2. Decomposition of the maximal affine subdiagrams into their components

The second of these has vertex set $S = \{n+2, n+4\}$. Let $T = \{1, 3, 4, \dots, n-1, n+1\}$. Once again T has type D_5 when $n = 6$, E_6 when $n = 7$, and E_7 when $n = 8$.

The remaining two dotted line subgraphs appear for all n , and they cannot be checked using the same quick method as the other two. The first of these has vertex set $S = \{1, n+1\}$. The two associated vectors are

$$\begin{aligned} e_1 &= -v_1 + v_2 \\ e_{n+1} &= 2v_0 + 5v_1 \end{aligned}$$

so a vector $v \in K_S$ has the form

$$v = av_0 + 2a(v_1 + v_2) + \sum_{i=3}^n k_i v_i$$

Since $K_S \subset K$, $(v, e_i) \leq 0$ for all e_i . A consequence of this is that $2a \geq k_3 \geq \dots \geq k_n \geq 0$.

When $n = 2$, $(v, e_2) \leq 0 \Rightarrow a \geq 0$, and $(v, e_4) \leq 0 \Rightarrow a \leq 0$, so $a = 0$.

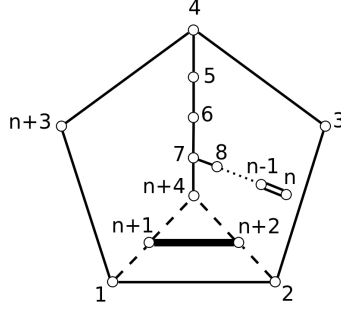


FIGURE 2. Coxeter diagram of the polyhedron in dimension 9 with the first four vectors found by Vinberg's algorithm

When $n \geq 3$, $(v, e_{n+3}) \leq 0 \Rightarrow k_3 + k_4 \leq -a$. Since $2a \geq 0$, $a \geq 0$ so $-a \leq 0$. Also, since $k_3 \geq k_4 \geq 0$, $k_3 + k_4 \geq 0$. Therefore $a = 0$, and so if $v \in K_S$, $v = 0$.

The other dotted line subgraph that appears for all n is the one with vertex set $S = \{2, n+2\}$. The associated vectors are

$$\begin{aligned} e_2 &= -v_2 + v_3 \\ e_{n+2} &= 3v_0 + 5v_1 + 5v_2 \end{aligned}$$

A vector $v \in K_S$ has the form

$$av_0 + bv_1 + (3a - b)(v_2 + v_3) + \sum_{i=4}^n k_i v_i$$

Since $K_S \subset K$, $(v, e_i) \leq 0$ for all e_i . In particular this holds for $i = 1, \dots, n$, and we have $b \geq 3a - b \geq k_4 \geq \dots \geq k_n \geq 0$. Since $b \geq 0$ and $3a - b \geq 0$, $a \geq 0$.

When $n = 2$, $(v, e_2) = 0 \Rightarrow b = 3a$, so $v = av_0 + 3av_1 + 2av_2$. Then $(v, e_3) \leq 0 \Rightarrow a \leq 0$, so it must be true that $a = 0$ in which case $v = 0$.

When $n \geq 3$, $(v, e_{n+3}) \leq 0 \Rightarrow a + k_4 \leq 0$. Since $k_4 \geq 0$, $a \leq 0$. Therefore $a = 0$, and so if $v \in K_S$, $v = 0$.

This concludes the proof that the diagrams in figure 1 all describe hyperbolic polytopes of finite volume.

Proposition 3. *There are no finitely generated reflection groups associated to the quadratic form $-5x_0^2 + x_1^2 + \dots + x_n^2$ in n -dimensions for $n \geq 9$.*

Proof. The computation is very similar to the one McLeod does in the $p = 3$ case [2].

When $n \geq 9$, Vinberg's algorithm finds the same first 4 vectors as when $n = 8$. The diagram for the polytope with these four vectors is shown in Figure 2. Let $S = \{3, 4, 5, 6, 7, 8, n+3, n+4\}$. Let G_S be the subgraph with vertex set S . G_S has type \tilde{D}_7 . Removing all vertices with an edge to G_S leaves just G_S if $n = 9$, and if $n > 9$ leaves also the subgraph Y with vertex set $\{10, \dots, n\}$. Y is of type B_{n-9} ($B_1 = A_1$, $B_2 = I_2(4)$). In order for the algorithm to eventually terminate, there must be some set T with $S \subset T \subset \{1, \dots, n+4\}$ such that G_T has rank $n - 1$ and G_S is a component of T .

Since G_S would be a component of G_T , any vertex i in $T \setminus S$ could have no edge to any vertex in S . The vector corresponding to such a vertex would have zero inner

product with the vectors labeled by elements of S . This imposes some conditions on $e_i = \sum_{\ell=0}^n k_\ell v_\ell$. The fact that $(e_i, e_j) = 0$ for $j = 3, 4, 5, 6, 7, 8$ implies that $k_3 = k_4 = \dots = k_9$. Also $(e_i, e_{n+3}) = 0$, so

$$5k_0 = 2k_1 + k_2 + 2k_3$$

and $(e_i, e_{n+4}) = 0$, so

$$5k_0 = k_1 + k_2 + 5k_3$$

Combine these two equations to get $k_1 = 3k_3$ and $k_0 = \frac{k_2+8k_3}{5}$, so that

$$e_i = \frac{k_2 + 8k_3}{5}v_0 + 3k_3v_1 + k_2v_2 + k_3(v_3 + \dots + v_9) + \sum_{j=10}^n k_jv_j$$

Then

$$5(e_i, e_i) = 4(k_2 - 2k_3)^2 + 5 \sum_{j=10}^n k_j^2$$

By the crystallographic condition, $5(e_i, e_i)$ must be 5, 10, 25, or 50. Subtracting 4 times a square from any of these cannot yield a non-negative multiple of 5. Therefore if this equation is to be satisfied, it must be that $k_2 - 2k_3 = 0$, meaning e_i has the form

$$e_i = 2k_3v_0 + 3k_3v_1 + 2k_3v_2 + k_3(v_3 + \dots + v_9) + \sum_{i=10}^n k_i^2$$

and

$$(e_i, e_i) = \sum_{i=10}^n k_i^2$$

When $n = 9$, this means Vinberg's algorithm won't find any vectors with no edge to G_S .

Suppose $n > 9$. If $(e_i, e_i) = 5$ or 10, k_i is divisible by 5 for $i > 0$, so $k_{10}^2 \geq 25$. But then if $k_{10} \neq 0$ already the norm is larger than 10, so (e_i, e_i) cannot be 5 or 10. If $(e_i, e_i) = 1$, $k_{10} = 1$ and $k_i = 0$ for $i > 10$. If $(e_i, e_i) = 2$, $k_{10} = k_{11} = 1$ and $k_i = 0$ for $i > 11$. There is a family of vectors that the algorithm could find:

$$\begin{aligned} &2av_0 + 3av_1 + 2av_2 + a(v_3 + \dots + v_9) + v_{10} \\ &2av_0 + 3av_1 + 2av_2 + a(v_3 + \dots + v_9) + v_{10} + v_{11} \end{aligned}$$

Any two of these have positive inner product with each other for any choice of values for a , so Vinberg's algorithm will only ever find one of them. Even if that one has an edge to the subdiagram Y , making a \tilde{B}_{n-9} , the combined rank with G_S is only $n - 2$ which is not enough to satisfy condition 1 of the finite volume criterion. □

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